Three contrasts between two senses of *coherence*

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Call an agent's choices *coherent* when they respect *simple dominance* relative to a (finite) partition.

$\Omega = \{\omega_1, \ldots, \omega_n\}$ is a finite partition of the sure event: a set of *states*.

Consider two acts A_1, A_2 defined by the their outcomes relative to Ω.

	ω_1	ω_2	ω_3	…	ω_n
A_1	o_{11}	o_{12}	o_{13}	…	o_{1n}
A_2	o_{21}	o_{22}	o_{23}	…	o_{2n}

Suppose the agent can compare the desirability of different outcomes at least within each state, and, for each state ω_j, outcome o_{2j} is (strictly) preferred to outcome o_{1j}, $j = 1, \ldots, n$. Then A_2 simply dominates A_1 with respect to Ω.

- ***Coherence*: When A_2 simply dominates A_1 in some finite partition, then A_1 is inadmissible in any choice problem where A_2 is feasible.**

Background on de Finetti's two senses of *coherence*

De Finetti (1937, 1974) developed two senses of *coherence* (*coherence*$_1$ and *coherence*$_2$), which he extended also to infinite partitions.

Let $\Omega = \{\omega_1, \ldots, \omega_n, \ldots\}$ be a countable partition of the sure event: a finite or denumerably infinite set of *states*.

Let $\chi = \{X_i\colon \Omega \to \Re;\ i = 1, \ldots\}$ be a countable class of (bounded) real-valued random variables defined on Ω.

That is, $X_i(\omega_j) = r_{ij}$ and for each $X \in \chi$, $-\infty < inf_\Omega X(\omega) \leq sup_\Omega X(\omega) < \infty$.

Consider random variables as acts, with their associated outcomes.

	ω_1	ω_2	ω_3	$\ldots$	ω_n	$\ldots$
X_1	r_{11}	r_{12}	r_{13}	$\ldots$	r_{1n}	$\ldots$
X_2	r_{21}	r_{22}	r_{23}	$\ldots$	r_{2n}	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_i	r_{i1}	r_{i2}	r_{i3}	$\ldots$	r_{in}	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

***Coherence*$_1$: de Finetti's (1937) the 0-sum *Prevision Game* – wagering.**

The players in the *Prevision Game*:

- **The *Bookie* – who, for each random variable X in χ announces a *prevision* (a *fair price*), $P(X)$, for buying/selling units of X.**
- **The *Gambler* – who may make finitely many (non-trivial) contracts with the *Bookie* at the *Bookie*'s announced prices.**

For an individual contract, the *Gambler* fixes a real number α_X, which determines the contract on X, as follows.

In state ω, the contract has an *outcome* to the *Bookie* (and opposite outcome to the *Gambler*) of $\alpha_X[X(\omega) - P(X)] = O_\omega(X, P(X), \alpha_X)$.

When $\alpha_X > 0$, the *Bookie* buys α_X-many units of X from the *Gambler*.
When $\alpha_X < 0$, the *Bookie* sells α_X-many units of X to the *Gambler*.

The *Gambler* may choose finitely many non-zero ($\alpha_X \neq 0$) contracts.

The *Bookie*'s net *outcome* in state ω is the sum of the payoffs from the finitely many non-zero contracts: $\sum_{X\in\chi} O_\omega(X, P(X), \alpha_X) = O(\omega)$.

***Coherence*$_1$: The *Bookie*'s previsions $\{P(X): X \in \chi\}$ are *coherent*$_1$**

provided that there is no strategy for the *Gambler* that results in a sure

(uniform) net loss for the *Bookie*.

$$\neg\exists(\{\alpha_{X1}, \ldots, \alpha_{Xk}\}, \varepsilon > 0), \forall\omega\in\Omega \sum_{X\in\chi} O_\omega(X, P(X), \alpha_X) \leq -\varepsilon.$$

Otherwise, the *Bookie*'s previsions are *incoherent*$_1$.

The net outcome *O* is just another random variable.

The *Bookie*'s *coherent*$_1$ previsions do not allow the *Gambler* contracts where the *Bookie*'s net-payoff is uniformly dominated by *Abstaining*.

	ω_1	ω_2	ω_3	…	ω_n	…
O	$O(\omega_1)$	$O(\omega_2)$	$O(\omega_3)$	…	$O(\omega_n)$	…
Abstain	0	0	0	…	0	…

***Coherence$_2$*: de Finetti's (1974) *Forecasting Game* (with Brier Score)**

There is only the one player in the *Forecasting Game*, the *Forecaster*.

- **The *Forecaster* – who, for random variable X in χ announces a real-valued *forecast* $F(X)$, subject to a squared-error loss outcome.**

In state ω, the *Forecaster* is penalized $-[X(\omega) - F(X)]^2 = O_\omega(X, F(X))$.

The *Forecaster*'s net score in state ω from forecasting finitely variables

$\{F(X_i)$: $i = 1, \ldots, k\}$ is the sum of the k-many individual losses

$$\sum_{i=1}^{k} O_\omega(X, F(X_i)) = \sum_{i=1}^{k} -[X_i(\omega) - F(X_i)]^2 = O(\omega).$$

Coherence$_2$: The *Forecaster*'s forecasts $\{F(X): X \in \chi\}$ are *coherent*$_2$ provided that there is no finite set of variables, $\{X_1, \ldots, X_k\}$ and set of rival forecasts $\{F'(X_1), \ldots, F'(X_k)\}$ that yields a uniform smaller net loss for the *Forecaster* in each state.

$$\neg\exists(\{F'(X_1), \ldots, F'(X_k)\}, \varepsilon > 0), \forall\omega\in\Omega$$

$$\sum_{i=1}^{k} -[X_i(\omega) - F(X_i)]^2 \leq \sum_{i=1}^{k} -[X_i(\omega) - F'(X_i)]^2 - \varepsilon.$$

Otherwise, the *Forecaster*'s forecasts are *incoherent*$_1$.

The *Forecaster*'s *coherent*$_2$ previsions do not allow rival forecasts that uniformly dominate in Brier Score (i.e., squared-error).

	ω_1	ω_2	ω_3	…	ω_n	…
O	$O(\omega_1)$	$O(\omega_2)$	$O(\omega_3)$	…	$O(\omega_n)$	…
O'	$O'(\omega_1)$	$O'(\omega_1)$	$O'(\omega_1)$	…	$O'(\omega_1)$	…

Theorem **(de Finetti, 1974):**

A set of previsions {*P*(*X*)} are *coherent*$_1$.

if and only if

The same *forecasts* {*F*(*X*): *F*(*X*) = *P*(*X*)} are coherent$_2$.

if and only if

There exists a (finitely additive) probability P such that these quantities are the P-Expected values of the corresponding variables

$$E_P[X] = F(X) = P(X).$$

***Corollary*: When the variables are 0-1 indicator functions for events, *A*,**

$$I_A(\omega) = 1 \text{ if } \omega \in A \text{ and } I_A(\omega) = 0 \text{ if } \omega \notin A,$$

then de Finetti's theorem asserts:

Coherent prices/forecasts must agree with the values of a (finitely additive) probability distribution over these same events.

Otherwise, they are incoherent.

***Example*:**

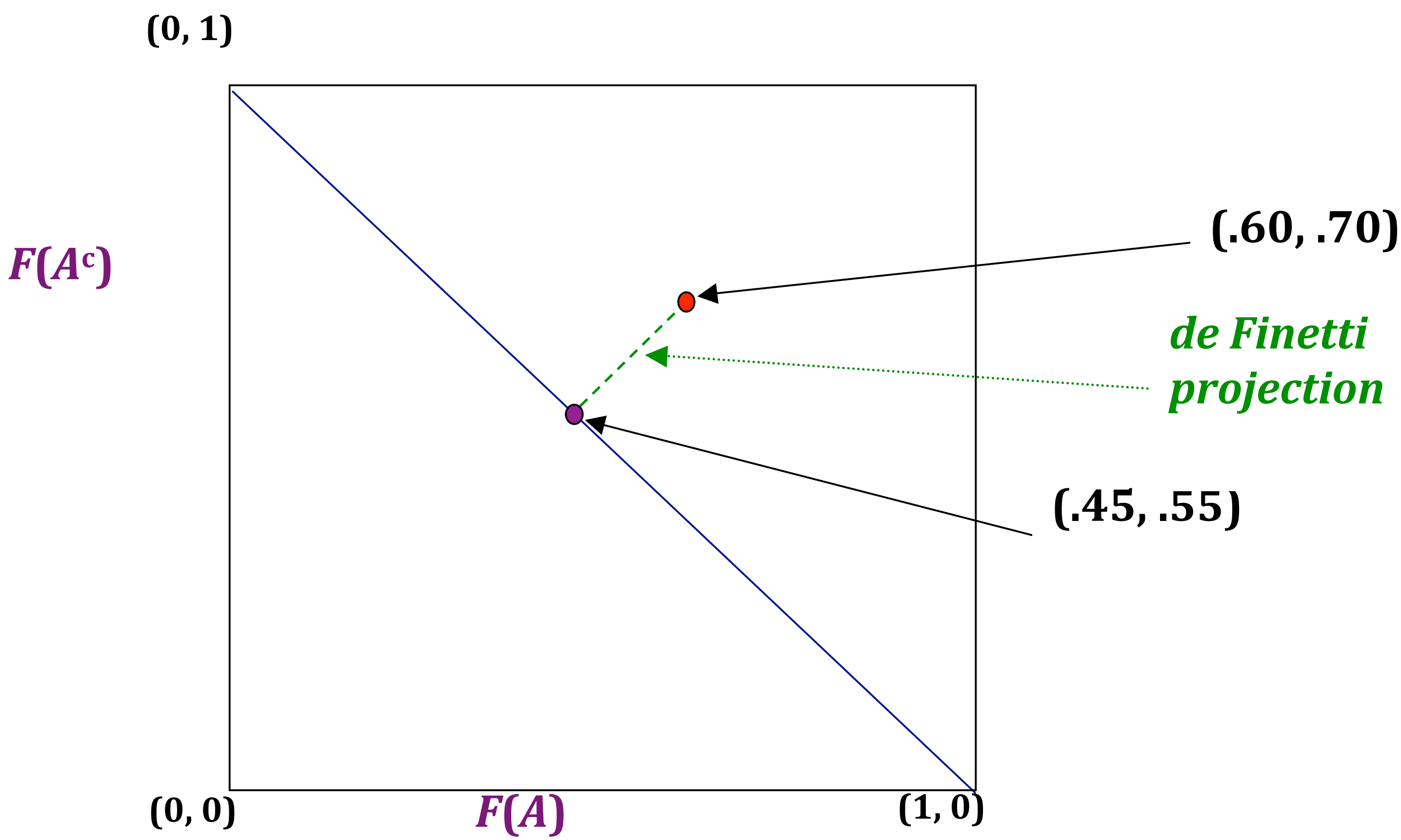
***A Bookie*'s two previsions, {$P(A)=.6$; $P(A^c)=.7$}, are incoherent$_1$**

The *Bookie* has overpriced the two variables.

A *Book* is achieved against these previsions with the *Gambler*'s strategy $\alpha_A = \alpha_{A^c} = 1$, requiring the *Bookie* to buy each variable at the announced price.

The net payoff to the *Bookie* is -0.3 regardless which state ω obtains.

In order to see that these are also *incoherent*$_2$ forecasts, review the following diagram


(0, 1)
F(A^c)
(.60, .70)
de Finetti
projection
(.45, .55)
(0, 0)
F(A)
(1, 0)

If the forecast previsions are not coherent$_1$, they lie outside the probability simplex. Project these incoherent$_1$ forecasts into the simplex. As in the *Example*, (.60, .70) projects onto the coherent$_1$ previsions depicted by the point (.45, .55). By elementary properties of Euclidean projection, the resulting coherent$_1$ forecasts are closer to each endpoint of the simplex. Thus, the projected forecasts have a dominating Brier score regardless which state obtains. This establishes that the initial forecasts are incoherent$_2$. Since no coherent$_1$ forecast set can be so dominated, we have coherence$_1$ of the previsions if and only coherence$_2$ of the corresponding forecasts.

Background on Coherence and Elicitation

De Finetti's interest in *coherence*$_2$, avoiding dominated forecasts under squared-error loss (Brier Score), was prompted by an observation due to Brier (1950).

<u>Theorem</u> **(Brier, 1950) A SEU forecaster whose forecasts are scored by squared error loss in utility units, (uniquely) maximizes expected utility by announcing her/his expected value for each forecast variable.**

- **Brier Score is a (*strictly*) *proper scoring rule*.**

That is, squared error loss provides the incentives for an SEU forecaster to be entirely straightforward with her/his forecasts.

A moment's reflection establishes that wagering, as in the *Prevision Game*, does *not* ensure the right incentives are present for the *Bookie* always to announce her/his expected $\mathrm{E}_P(X)$ value as the "fair price" $P(X)$ for variable X.

Suppose that the *Bookie* has an opinion about the *Gambler*'s fair betting odds on an event, A.

Suppose the Bookie believes: $\mathrm{E}_P[I_A] < \mathrm{E}_P[I_A]$.

Then it is strategic for the Bookie to announce a prevision:

$$\mathrm{E}_P[I_A] < P(A) < \mathrm{E}_P[I_A].$$

The 1st contrast between two senses of coherence:
infinitely many previsions/forecasts at once.

(1) Recall that de Finetti's coherence criteria require that the *Bookie*/*Forecaster* respects dominance only with respect to random variables created by *finite* combinations of fair-gambles/forecasts.

(2) Also, for infinite Ω, de Finetti restricted the dominance principle to require that the dominating option has *uniformly better* outcomes:

better in each state $\omega \in \Omega$ by at least some fixed amount, $\varepsilon > 0$.

Why these twin restrictions on the simple dominance principle?

The answer is because de Finetti (like, e.g., Savage) made room under the a Big Tent of coherent preferences for finitely (but not necessarily countably) additive probabilities.

***Example* 1 (de Finetti, 1949).**

Let Let $\Omega = \{\omega_1, \ldots, \omega_n, \ldots\}$ be a denumerably infinite partition of "equally probable" states. *Bookie*'s previsions are $P(\{\omega_i\}) = 0$, $i = 1, \ldots$.

The *Bookie* judges *fair* each gamble of the form $\alpha_i(I\omega_i - 0)$.
Thus, *Bookie*'s personal probability is strongly finitely additive, as

$$0 = \sum_i P(\{\omega_i\}) < P(\cup_i \{\omega_i\}) = P(\Omega) = 1.$$

These are coherent$_1$ previsions, by de Finetti's *Theorem*.

However, if the *Gambler* is allowed to engage in more than finitely many contracts at a time, even assuring that the net-outcome is finite and bounded in every state, there is a simple strategy that causes the *Bookie* to suffer a uniform (sure) loss.

Set $\alpha_i = -1$. Then, $\forall \omega \in \Omega$, $\Sigma_i \alpha_i(I\omega_i(\omega) - 0) = -\Sigma_i I\omega_i(\omega) = -1$.

De Finetti noted: a sure-loss obtains in this fashion if and only if the *Bookie*'s previsions are not countably additive.

However, no such failure of dominance results by combining infinitely many forecasts, provided that the *Forecaster*'s expected score is finite.

Assume that expectations for sums of the random variables to be forecast, and also for their squares, are *absolutely convergent*:

$$\mathbf{E_P[\textstyle\sum_i |X_i|\,] \leq V < \infty \qquad (1)}$$

$$\mathbf{E_P[\textstyle\sum_i X_i^2] \leq W < \infty. \qquad (2)}$$

***Proposition* 1: Let $\chi = \{X_i, i = 1\ldots\}$ be a class of variables and P a finitely additive probability satisfying conditions (1) and (2), with coherent$_2$ forecasts $E_P[X_i] = p_i$.**

There does not exist a set of real numbers $\{q_i\}$ such that

$$\forall\omega \in \Omega,\ \textstyle\sum_i(p_i - X_i(\omega))^2 - \sum_i(q_i - X_i(\omega))^2 > 0.$$

***Corollary*: When conditions (1) and (2) obtain, the infinite sum of Brier scores applied to the infinite set of forecasts $\{p_i\}$ is a strictly proper scoring rule.**

***Proposition* 1 and its *Corollary* establish that the two senses of coherence are *not* equivalent when considering finitely additive probabilities and infinite sets of previsions/forecasts.**

Assume the finiteness conditions (1) and (2).

Coherence$_1$, associated with the *Prevision Game*, depends upon the requirement that only finitely many *fair* contracts may be combined at once while permitting finitely (but not countably) additive probabilities to be *coherent*.

Coherence$_2$, associated with the *Forecasting Game*, has no such restrictions for combining infinitely many forecasts. Moreover, Brier score retains its status as a strictly proper scoring rule even when infinitely many variables are forecast simultaneously.

- **Contrast #1 favors Coherence$_2$ over Coherence$_1$!**

The 2nd contrast between two senses of coherence: *moral hazard*.

Consider the following case of simple dominance between two acts.

	ω_1	ω_2
A_1	*3*	*1*
A_2	*4*	*2*

Act A_2 simply dominates act A_1.

However, if there is *moral hazard* – act-state probabilistic dependence, then A_1 may maximize subjective (conditional) expected utility, not A_2.

For example, consider circumstances where $P(\omega_i \mid A_i) \approx 1$, for $i = 1, 2$.

Then, $$SE_{A_1}U(A_1) \approx 3 > 2 \approx SE_{A_2}U(A_2).$$

The agent strictly prefers A_1 over A_2.

- **With moral hazard, *simple dominance* is not compelling.**

However, there is a more restrictive version of dominance that is robust against the challenge of *moral hazard*.

Consider two acts A_1, A_2 defined by the their outcomes relative to Ω.

	ω_1	ω_2	ω_3	…	ω_n
A_1	o_{11}	o_{12}	o_{13}	…	o_{1n}
A_2	o_{21}	o_{22}	o_{23}	…	o_{2n}

Suppose the agent can compare the desirability of *all* pairs of different outcomes. The agent can compare outcome o_{ij} and o_{kl} for all pairs, and ranks them in some (strict) weak order $\prec$.

Say that A_2 *robustly dominates* A_1 with respect to Ω when,

$$\prec\text{-}max_{\Omega}\{o_{1j}\} \prec \prec\text{-}min_{\Omega}\{o_{1j}\}.$$

The $\prec$-best of all possible outcomes under A_1 is strictly $\prec$-dispreferred to the $\prec$-worst of all possible outcomes under A_2

- **It is immediate that *Robust Dominance* accords with SEU even in the presence of (arbitrary) moral hazards.**

***Proposition* 2: Each instance of incoherence$_1$, but not of incoherence$_2$, is a case of *Robust Dominance*.**

***Abstaining* is strictly preferred to *Book* regardless of moral hazard.**

But the same incoherent$_2$ forecast, though dominated in Brier score by a rival forecast, may have greater expected utility than that dominating rival forecast when there is moral hazard connecting forecasting and the states forecast.

***Example 2*: The *bookie* is asked for a pair of *fair* betting odds, one for an event R and one for its complement R^c.**

The same agent forecasts the same pair of events subject to Brier score.

The pair $P(R) = .6$ and $P(R^c) = .9$ are incoherent in both of de Finetti's senses, since $P(R) + P(R^c) = 1.5 > 1.0$.

For demonstrating incoherence$_1$, the *gambler* chooses $\alpha_R = \alpha_{R^c} = 1$, which produces a sure-loss of -0.5 for the *bookie*.

That is, $1(I_R(\omega) - .6) + 1(I_{R^c}(\omega) - .9) = -0.5 < 0$ in each state, $\omega \in \Omega$.

Hence, *Abstaining* from betting, with a constant payoff 0, *robustly dominates* the sum of these two *fair* bets in the partition by states Ω.

The *Forecaster* announces $F(R) = .60$ and $F(R^c) = .90$.

For demonstrating incoherence$_2$, consider the rival coherent forecasts

$$Q(R) = .35 \text{ and } Q(R^c) = .65,$$

the de Finetti projection of the point (.6, .9) into the coherent simplex.

For states $\omega \in R$,

the Brier score for the two F-forecasts is $(1-.6)^2 + (0-.9)^2 = .970$

the Brier score for the rival Q-forecasts is $(1-.35)^2 + (0-.65)^2 = .845$.

For states $\omega \notin R$,

the Brier score for the two F-forecasts is $(0-.6)^2 + (1-.9)^2 = .370$

the Brier score for the rival Q-forecasts is $(0-.35)^2 + (1-.65)^2 = .245$.

- **The Brier score for the rival Q-forecasts (.35, .65) *simply dominates*, but does <u>not</u> *robustly dominate* the Brier score for the F-forecasts (.6, .9) in the partition by states Ω.**

Consider a case of moral hazard in betting, or in forecasting, as before:

Let the moral hazards associated with betting be any which way at all!

Conditional on making the incoherent$_2$ F-forecasts (.6, .9),
the agent's conditional probability for event R^c is nearly 1.

But conditional on making the rival (coherent) *Q*-forecasts (.35, .65) the agent's conditional probability for *R* is nearly 1.

Then it remains the case that given the incoherent$_1$ pair of betting odds (.6, .9), the *bookie* has a negative conditional expected utility of -0.5 when the *gambler* chooses $\alpha_R = \alpha_{R^c} = 1$, regardless the moral hazards relating betting with the events wagered.

Offering those incoherent$_1$ betting odds remains strictly dispreferred to *Abstaining*, which has conditional expected utility 0 even in this case of extreme moral hazard. *Abstaining* robustly dominates a *Book*.

However, with the assumed moral hazards for forecasting:

The conditional expected loss under Brier score given the incoherent$_2$ *F*-forecast pair (.6, .9) is nearly .370.

The conditional expected loss under Brier score given the rival coherent and dominating *Q*-forecast pair (.35, .65) is nearly .845.

That is, though the rival coherent$_2$ *Q*-forecast pair (.35, .65) simply dominates the incoherent$_2$ *F*-forecast pair (.6, .9) in combined Brier score, as this is <u>not</u> a case of *robust dominance*, with moral hazard it may be the that incoherent$_2$ forecast is strictly preferred.

With these moral hazards, each rival *Q′*-forecast that simply dominates the incoherent$_2$ *F*-forecast pair (.6, .9) has <u>lower</u> conditional expected utility and is dispreferred to the incoherent$_2$ *F*-forecasts.

- **Contrast #2 favors Coherence$_1$ over Coherence$_2$!**

A 3rd contrast between two senses of coherence: *state-dependent utility*.

Assume that there are no *moral hazards*:
states are probabilistically independent of acts.

Begin with a *trivial* result about *equivalent* SEU representations.

Suppose an SEU agent's $\succ$ preferences over acts on $\Omega = \{\omega_1, \ldots, \omega_n\}$ is represented by prob/state-dependent utility pair (P; U_j: $j = 1, \ldots, n$).

	ω_1	ω_2	ω_3	…	ω_n
A_1	o_{11}	o_{12}	o_{13}	…	o_{1n}
A_2	o_{21}	o_{22}	o_{23}	…	o_{2n}

$$A_2 \succ A_1 \quad \textit{if and only if} \quad \sum_j P(\omega_j)U_j(o_{2j}) > \sum_j P(\omega_j)U_j(o_{1j}).$$

Let Q be a probability on Ω that agrees with P on null events:

$$P(\omega) = 0 \textit{ if and only if } Q(\omega) = 0.$$

Let U'_j be defined as c_jU_j, where $c_j = P(\omega_j)/Q(\omega_j)$.

***(Trivial Result) Proposition* 3:**

$$(P; U_j) \text{ represents } \succ \quad \textit{if and only if} \quad (Q; U'_j) \text{ represents } \succ.$$

Example 3: The de Finetti _Prevision Game_ for a single event G.
For simplicity, let $\Omega = \{\omega_1, \omega_2\}$ with $G = \{\omega_1\}$.

Suppose that, when betting in US dollars, $, the _Bookie_ posts fair odds $P^{\$}(G) = 0.5$, so that she/he judges as _fair_ contracts of the form

$$\$\alpha(I_G - .5).$$

Suppose that, when betting in Euros, €, the same _Bookie_ posts fair odds $P^{€}(G) = 5/11 = 0.\overline{45}$, so that she/he judges as _fair_ contracts of the form

$$€\alpha(I_G - 5/11).$$

- **Is the _Bookie_ coherent$_1$? _Answer_: YES!**

- **Why do the _Bookie_'s previsions depend upon the currency?**

Answer: Because the _Bookie_'s currency valuations are state-dependent!

In state ω_1	**In state ω_2**
€1 $\equiv$ \$1.25	**€1 $\equiv$ \$1.50**

	ω_1	ω_2
D_1	**\$1**	**\$0**
D_2	**\$0**	**\$1**

The *Bookie* is indifferent between acts D_1 and D_2 since she/he has \$-fair-betting rates of ½ on each state.

So, then the *Bookie* is indifferent between acts E_1 and E_2

	ω_1	ω_2
E_1	**€0.80**	**€0**
E_2	**€0**	**€0.67**

which mandates €-fair betting rates of 5/11 : 6/11 on $\omega_1 : \omega_2$.

***Aside*: The *Bookie* has a fair currency exchange rate of €1 $\equiv$ \$1.375.**

But by the *Trivial Result* – there is no way to separate fair-odds (degrees of belief) from currency (utility values) based on coherent betting odds!

One ($\$P$, U_j) pair uses a state-independent utility for Dollars and a state dependent utility for Euros.

One (€Q; U'_j) pair uses a state-independent utility for Euros and a state dependent utility for Dollars.

- **Fixing coherent personal probabilities in the *Prevision Game* does *not* allow a separation of beliefs from values.**

What is the situation in the Forecasting Game?

What happens to the agent's coherent$_2$ forecasts when Brier score is made operational in Dollar units, rather than in Euro units?

Does propriety of squared-error loss resolve which is the *Forecaster*'s *real* degrees of belief

The answer is that the *Trivial Result* applies to *all* decisions over a set of acts, including those in the *Forecasting Game*.

When scored in Dollars, the coherent$_2$ *Forecaster* will maximize expected utility by offering forecasts corresponding to the (\$$P$, U_j) pair, which uses a state-independent utility for Dollars and a state dependent utility for Euros.

When scored in Euros, the coherent$_2$ *Forecaster* will maximize expected utility by offering forecasts corresponding to the (€Q; U'_j) pair, which uses a state-independent utility for Euros and a state dependent utility for Dollars.

Neither the *Prevision Game* nor the *Forecasting Game* solves the problem posed by the *Trivial Result*, the problem of separating beliefs from values based on preferences over acts.

- **Contrast #3 favors *neither* Coherence$_1$ nor Coherence$_2$. Both fail !!**

Summary

In three different contrasts between de Finetti's two senses of coherence, we have these varying results:

#1: Coherence$_1$ – *Previsions* immune to Book – <u>does not</u>, but
Coherence$_2$ – *Forecasting* subject to Brier score – <u>does</u>

permit the infinite combinations of previsions/forecasts that are separately coherent when these arise from a (merely) f.a. probability.

#2: Coherence$_2$ – *Forecasting* subject to Brier score – <u>does not</u>, but
Coherence$_1$ – *Previsions* immune to Book – <u>does</u>

permit arbitrary cases of *moral hazard*.

#3: Neither Coherence$_1$ – *Previsions* immune to Book,
Nor Coherence$_2$ – undominated *Forecasts* according to Brier score,

solves the challenge posed by the *Trivial Result* for separating beliefs from values based on preferences over acts.

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